

Complete description of polarization effects in the nonlinear Compton scattering

I. Circularly polarized laser photons

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October 29, 2003

Abstract

We consider emission of a photon by an electron in the field of a strong laser wave. Polarization effects in this process are important for a number of physical problems. We discuss a probability of this process for circularly polarized laser photons and for arbitrary polarization of all other particles. We obtain the complete set of functions which describe such a probability in a compact covariant form. Besides, we discuss an application of the obtained formulas to the problem of $e \rightarrow \gamma \gamma$ conversion at $\gamma\gamma$ and γe colliders.

1 Introduction

The Compton scattering

$$e(p) + \gamma(k) \rightarrow e(p') + \gamma(k') \quad (1)$$

is one of the first processes calculated in quantum electrodynamics at the end of 20's. The analysis of its polarization effects is now included in text-books (see, for example, [1] §87). Nevertheless, the complete description of cross sections with the polarization of both initial and final particles has been considered in detail only recently (see [2, 3, 4] and literature therein). One of the interesting application of this process is related to collisions of ultra-relativistic electrons with the beam of polarized laser photons. In this case the Compton effect is a basic process at obtaining high-energy photons for contemporary experiments in nuclear physics (photonuclear reactions with polarized photons) and for future $\gamma\gamma$ and γe colliders [5]. The importance of the particle polarization is clearly seen from the fact that the number of photons with maximum energy is nearly doubled when helicities of the initial electron and photon are opposite.

With the growth of the laser field intensity, an electron collides (with an essential probability) with n laser photons simultaneously,

$$e(q) + n \gamma_L(k) \rightarrow e(q') + \gamma(k'), \quad (2)$$

thus the Compton scattering becomes nonlinear. Such a process with absorption of $n = 1, 2, 3, 4$ laser photons was observed in recent experiment at SLAC [6]. The polarization properties of process (2) are important for a number of problems, for example, for obtaining highly polarized electrons and positrons beams [10], for a laser beam cooling [7] and, especially, for future $\gamma\gamma$ and γe colliders. In the latter case the nonlinear Compton scattering must be taken into account at simulation of the processes in the conversion region. For comprehensive simulation, including processes of multiple electron scattering, one has to know not only the differential cross section of the nonlinear Compton scattering, but energy, angles and polarization of final photons and electrons as well. The method for calculation of such cross sections was developed by Nikishov and Ritus [8]. It is based on the exact solution of the Dirac equation in the external electromagnetic plane wave. Some particular polarization properties of this process for the circularly polarized laser photons were considered in [9, 10, 11, 12] and have already been included in the existing simulation codes [11, 13]. In the present paper we give the complete description of the nonlinear Compton scattering for the case of circularly polarized laser photons and arbitrary polarization of all other particles. The case of the linearly polarized laser photons will be considered in the next paper. We follow the method of Nikishov and Ritus in the form presented in [1] §101.

In the next section we describe the kinematics. The cross section in the invariant form, including polarization of all particles, is obtained in Sect. 3. In Sect. 4 we considered in detail the polarization of final particles in the frame of reference relevant for $\gamma\gamma$ and γe colliders. In Sect. 5 we summarized our results and compare them with those known in the literature. In Appendix we give a comparison of the obtained cross section in the limit of weak laser field with the cross section for the linear Compton scattering.

2 Kinematics

2.1 Parameter of nonlinearity

Let us consider the interaction of an electron with a monochromatic plane wave described by 4-potential A_μ (the corresponding electric and magnetic fields are \mathbf{E} and \mathbf{B} , a frequency is ω) and let F be the root-mean-squared field strength,

$$F^2 = \langle \mathbf{B}^2 \rangle = \langle \mathbf{E}^2 \rangle.$$

The invariant parameter describing the intensity of the laser field (the parameter of nonlinearity) is defined via the mean value of squared 4-potential:

$$\xi = \frac{e}{mc^2} \sqrt{-\langle A_\mu A^\mu \rangle}, \quad (3)$$

where e and m is the electron charge and mass, c is the velocity of light. The origin of this parameter can be explained as follows. The electron oscillates in the transverse direction under influence of the force $\sim eF$ and for the time $\sim 1/\omega$ acquires the transverse momentum $\sim p_\perp = eF/\omega$, thus for the longitudinal motion its effective mass becomes $m_* = \sqrt{m^2 + (p_\perp/c)^2}$. The ratio of the momentum p_\perp to mc is the natural dimensionless parameter

$$\xi = \frac{eF}{mc\omega}. \quad (4)$$

This parameter can be also expressed via the density n_L of photons in the laser wave:

$$\xi^2 = \left(\frac{eF}{m\omega c} \right)^2 = \frac{4\pi\alpha\hbar}{m^2 c \omega} n_L, \quad (5)$$

where \hbar is the Plank constant.

From the classical point of view, the oscillated electron emits harmonics with frequencies $n\omega$, where $n = 1, 2, \dots$. Their intensities at small ξ is proportional to $(\mathbf{E}^2)^n \propto \xi^{2n}$, the polarization properties of these harmonics depends on the polarizations of the laser wave and the initial electron¹. From the quantum point of view, this radiation can be described as the nonlinear Compton scattering with absorbtion of n laser photons. When describing such a scattering, we have to take into account that in a laser wave 4-momenta p and p' of the free initial and final electrons are replaced by the 4-quasi-momenta q and q' (similar to the description of the particle motion in a periodic potential field in non-relativistic quantum mechanics),

$$\begin{aligned} q &= p + \xi^2 \frac{m^2 c^2}{2pk} k, \quad q' = p' + \xi^2 \frac{m^2 c^2}{2p'k} k, \\ q^2 &= (q')^2 = (1 + \xi^2) m^2 c^2 \equiv m_*^2 c^2. \end{aligned} \quad (6)$$

In particular, the energy of the free incident electron E is replaced by the quasi-energy

$$cq_0 = E + \xi^2 \frac{m^2 c^2}{2pk} \hbar\omega. \quad (7)$$

2.2 Invariant variables

As a result, we deal with the reaction (2) for which the conservation law reads

$$q + n k = q' + k'. \quad (8)$$

From this it follows that all the kinematic relations, which occur for the linear Compton scattering, will apply to the process considered here if the electron momenta p and p' are replaced by the quasi-momenta q and q' and the incident photon momentum k by the 4-vector nk . Since $qk = pk$, we can use the same invariant variables as for the linear Compton scattering (compare [3]):

$$x = \frac{2pk}{m^2 c^2}, \quad y = \frac{kk'}{pk}. \quad (9)$$

Moreover, many kinematic relations can be obtained from the previous ones by the replacement: $\omega \rightarrow n\omega$, $m \rightarrow m_*$. In particular, we introduce convenient combinations analogous to those used in the linear Compton scattering

$$s_n = 2\sqrt{r_n(1 - r_n)}, \quad c_n = 1 - 2r_n, \quad (10)$$

¹The polarization of the first harmonic is related to the tensor of the second rank $\langle E_i E_j^* \rangle$, in this case one needs only three Stokes parameters. The polarization properties of higher harmonics are connected with the tensors of higher rank $\langle E_i E_j \dots E_k^* \rangle$, in this case one needs more parameters for their description. It is one of the reason why the nonlinear Compton effect had been considered only for 100% polarized laser beam mainly for circular or linear polarization.

where

$$r_n = \frac{y(1 + \xi^2)}{(1 - y)nx}. \quad (11)$$

It is useful to note that these invariants have a simple notion, namely

$$s_n = \sin \tilde{\theta}, \quad c_n = \cos \tilde{\theta}, \quad r_n = \sin^2(\tilde{\theta}/2), \quad (12)$$

where $\tilde{\theta}$ is the photon scattering angle in the frame of reference where the initial electron is at rest on average ($\mathbf{q} = 0$). Therefore,

$$0 \leq s_n, \quad r_n \leq 1, \quad -1 \leq c_n \leq 1. \quad (13)$$

The maximum value of the variable y for the reaction (2) is

$$y \leq y_n = \frac{nx}{nx + 1 + \xi^2}. \quad (14)$$

The value of y_n is close to 1 for large n , but for a given n it decreases with the growth of the parameter ξ . With this notation one can rewrite s_n in another form

$$s_n = \frac{2}{y_n(1 - y)} \sqrt{y(y_n - y)(1 - y_n)}, \quad (15)$$

from which it follows that

$$s_n \rightarrow 0 \quad \text{at} \quad y \rightarrow y_n \quad \text{or at} \quad y \rightarrow 0. \quad (16)$$

The usual notion of the cross section is not applicable for reaction (2) and the description of this reaction is usually given in terms of the probability of the process per second \dot{W} . However, for the procedure of simulation in the conversion region as well as for the simple comparison with the linear case, it is useful to introduce the “effective cross section” given by the definition

$$d\sigma = \frac{d\dot{W}}{j}, \quad (17)$$

where

$$j = \frac{(qnk)c^2}{q_0 n \hbar \omega} n_L = \frac{m^2 c^4 x}{2q_0 \hbar \omega} n_L$$

is the flux density of colliding particles. Contrary to the usual cross section, this effective cross section does depend on the laser beam intensity, i.e. on the parameter ξ . This cross section is the sum over harmonics, corresponding to the reaction (2) with a given number n of the absorbed laser photons²:

$$d\sigma = \sum_n d\sigma_n. \quad (18)$$

²In this formula and below the sum is over those n which satisfy the condition $y < y_n$, i.e. this sum runs from some minimal value n_{\min} up to $n = \infty$, where n_{\min} is determined by the equation $y_{n_{\min}-1} < y < y_{n_{\min}}$.

2.3 Invariant polarization parameters

The invariant description of the polarization properties of both the initial and the final photons can be performed in the standard way (see [1] §87). We define a pair of unit 4-vectors

$$e^{(1)} = \frac{N}{\sqrt{-N^2}}, \quad e^{(2)} = \frac{P}{\sqrt{-P^2}}, \quad (19)$$

where³

$$\begin{aligned} N^\mu &= \varepsilon^{\mu\alpha\beta\gamma} P_\alpha (k' - n k)_\beta K_\gamma, \quad P_\alpha = (q + q')_\alpha - \frac{(q + q')K}{K^2} K_\alpha, \quad K_\alpha = n k_\alpha + k'_\alpha, \\ \sqrt{-N^2} &= m^3 xy \frac{s_n}{r_n} \sqrt{1 + \xi^2}, \quad \sqrt{-P^2} = m \frac{s_n}{r_n} \sqrt{1 + \xi^2}. \end{aligned}$$

The 4-vectors $e^{(1)}$ and $e^{(2)}$ are orthogonal to each other and to 4-vectors k and k' ,

$$e^{(i)} e^{(j)} = -\delta_{ij}, \quad e^{(i)} k = e^{(i)} k' = 0; \quad i, j = 1, 2.$$

Therefore, they are fixed with respect to the scattering plane of the process.

Let ξ_j and ξ'_j be the Stokes parameters for the initial and final photon which are defined with respect to 4-vectors $e^{(1)}$ and $e^{(2)}$. In the considered case of 100% circularly polarized laser beam

$$\xi_1 = \xi_3 = 0, \quad \xi_2 = P_c = \pm 1, \quad (20)$$

where P_c is the degree of the circular polarization of the laser wave or the initial photon helicity. The parameters ξ'_j enter the Compton cross section, they describe *the detector polarization* which essentially represents the properties of the detector as selecting one or other polarization of the final photon.

Let ζ and ζ' be the polarization vectors of the initial and final electrons. They determine the electron-spin 4-vectors

$$a = \left(\frac{\zeta \mathbf{p}}{m}, \zeta + \mathbf{p} \frac{\zeta \mathbf{p}}{m(E + m)} \right), \quad a' = \left(\frac{\zeta' \mathbf{p}'}{m}, \zeta' + \mathbf{p}' \frac{\zeta' \mathbf{p}'}{m(E' + m)} \right) \quad (21)$$

and the mean helicity of the initial and final electrons

$$\lambda_e = \frac{\zeta \mathbf{p}}{2|\mathbf{p}|}, \quad \lambda'_e = \frac{\zeta' \mathbf{p}'}{2|\mathbf{p}'|}. \quad (22)$$

Now we have to define invariants, which describe the polarization properties of the initial and the final electrons. For the electrons it is a more complicated task than it is for the photons. For the linear Compton scattering, the relatively simple description was obtained in [2] using invariants which have a simple meaning in the centre-of-mass system. However, this frame of reference is not convenient for the description of the nonlinear Compton scattering since it has actually to vary with the change of the number of laser photons n .

Our choice is based on the experience obtained in [4] and [12]. We exploit two ideas. First of all, the 4-vector $e^{(1)}$ is orthogonal to 4-vectors k , k' , p and p' , therefore, the invariants $\zeta_1 = -a e^{(1)}$ and $\zeta'_1 = -a' e^{(1)}$ are the transverse polarizations of the initial and

³Below we use the system of units in which $c = 1$, $\hbar = 1$.

the final electrons perpendicular to the scattering plane. Further, it is not difficult to check that invariant $ak/(2mx)$ is the helicity of the initial electron in the frame of reference, in which the electron momentum \mathbf{p} is anti-parallel to the initial photon momentum \mathbf{k} . Analogously, the invariant $a'k/[(2mx(1-y))]$ is the helicity of the final electron in the frame of reference, in which momentum of the final electron \mathbf{p}' is anti-parallel to the initial photon momentum \mathbf{k} . It is important that this interpretation is valid for any number of the absorbed laser photons. Furthermore, we will show that for practically important case the latter frame of reference is almost coincides with the former frame of reference.

Taken into account that $ap = a'p' = 0$, we find two sets of units 4-vectors:

$$\begin{aligned} e_1 &= e^{(1)}, & e_2 &= -e^{(2)} - \frac{\sqrt{-P^2}}{m^2x}k, & e_3 &= \frac{1}{m} \left(p - \frac{2}{x}k \right); \\ e'_1 &= e^{(1)}, & e'_2 &= -e^{(2)} - \frac{\sqrt{-P^2}}{m^2x(1-y)}k, & e'_3 &= \frac{1}{m} \left(p' - \frac{2}{x(1-y)}k \right). \end{aligned} \quad (23)$$

These vectors satisfy the conditions:

$$e_i e_j = -\delta_{ij}, \quad e_j p = 0; \quad e'_i e'_j = -\delta_{ij}, \quad e'_j p' = 0. \quad (24)$$

It allows us to represent the 4-vectors a and a' in the following covariant form

$$a = \sum_{j=1}^3 \zeta_j e_j, \quad a' = \sum_{j=1}^3 \zeta'_j e'_j, \quad (25)$$

where

$$\zeta_j = -a e_j, \quad \zeta'_j = -a' e'_j. \quad (26)$$

The invariants ζ_j and ζ'_j describe completely the polarization properties of the initial electron and the detector polarization properties of the final electron, respectively. To clarify the meaning of invariants ζ_j , it is useful to note that

$$\zeta_j = \boldsymbol{\zeta} \mathbf{n}_j, \quad (27)$$

where the corresponding 3-vectors are

$$\mathbf{n}_j = \mathbf{e}_j - \frac{\mathbf{p}}{E+m} e_{j0} \quad (28)$$

with e_{j0} being a time component of 4-vector e_j defined in (23). Using properties (24) for 4-vectors e_j , one can check that

$$\mathbf{n}_i \mathbf{n}_j = \delta_{ij}. \quad (29)$$

As a result, the polarization vector $\boldsymbol{\zeta}$ has the form

$$\boldsymbol{\zeta} = \sum_{j=1}^3 \zeta_j \mathbf{n}_j. \quad (30)$$

Analogously, for the final electron the invariants ζ'_j can be presented as

$$\zeta'_j = \boldsymbol{\zeta}' \mathbf{n}'_j, \quad \mathbf{n}'_j = \mathbf{e}'_j - \frac{\mathbf{p}'}{E'+m} e'_{j0}, \quad \mathbf{n}'_i \mathbf{n}'_j = \delta_{ij}, \quad (31)$$

such that

$$\boldsymbol{\zeta}' = \sum_{j=1}^3 \zeta'_j \mathbf{n}'_j. \quad (32)$$

3 Cross section in the invariant form

3.1 General relations

The effective differential cross section can be presented in the following invariant form

$$d\sigma(\xi'_i, \zeta'_j) = \frac{r_e^2}{4x} \sum_n F^{(n)} d\Gamma_n, \quad d\Gamma_n = \delta(q + n k - q - k') \frac{d^3 k'}{\omega'} \frac{d^3 q'}{q'_0}, \quad (33)$$

where $r_e = \alpha/m$ is the classical electron radius, and

$$F^{(n)} = F_0^{(n)} + \sum_{j=1}^3 \left(F_j^{(n)} \xi'_j + G_j^{(n)} \zeta'_j \right) + \sum_{i,j=1}^3 H_{ij}^{(n)} \zeta'_i \xi'_j. \quad (34)$$

Here function $F_0^{(n)}$ describes the total cross section for a given harmonic n , summed over spin states of the final particles:

$$\sigma_n = \frac{r_e^2}{x} \int F_0^{(n)} d\Gamma_n. \quad (35)$$

Items $F_j^{(n)} \xi'_j$ and $G_j^{(n)} \zeta'_j$ in Eq. (34) describe the polarization of the final photons and the final electrons, respectively. The last items $H_{ij}^{(n)} \zeta'_i \xi'_j$ stand for the correlation of the final particles' polarizations.

From Eqs. (33), (34) it is possible to deduce *the polarization of the final photon and electron resulting from the scattering process itself*. We denote the Stokes parameters describing this polarization by $\xi_j^{(f)}$ to distinguish them from the detected polarization ξ'_j . Analogously, we denote by $\zeta_j^{(f)}$ the invariant parameters of this polarization for the final electron to distinguish them from the detector polarization parameters ζ'_j . According to the usual rules (see [1] §65), we obtain the following expression for the Stokes parameters of the final photon:

$$\xi_j^{(f)} = \frac{F_j}{F_0}, \quad F_0 = \sum_n F_0^{(n)}, \quad F_j = \sum_n F_j^{(n)}; \quad j = 1, 2, 3. \quad (36)$$

The polarization of the final electron is given by invariants

$$\zeta_j^{(f)} = \frac{G_j}{F_0}, \quad G_j = \sum_n G_j^{(n)}, \quad (37)$$

therefore, its polarization vector is

$$\boldsymbol{\zeta}^{(f)} = \sum_{j=1}^3 \frac{G_j}{F_0} \mathbf{n}'_j. \quad (38)$$

In the similar way, the polarization properties for a given harmonic n are described by:

$$\xi_{j(n)}^{(f)} = \frac{F_j^{(n)}}{F_0^{(n)}}, \quad \zeta_{j(n)}^{(f)} = \frac{G_j^{(n)}}{F_0^{(n)}}. \quad (39)$$

3.2 The results

We have calculated coefficients $F_j^{(n)}$, $G_j^{(n)}$ and $H_{ij}^{(n)}$ using the standard technic presented in [1] §101. All the necessary traces have been calculated using the package MATEMATIKA. In the considered case of 100 % circularly polarized ($P_c = \pm 1$) laser beam, almost all dependence on the parameter ξ accumulates in three functions:

$$\begin{aligned} f_n &\equiv f_n(z_n) = J_{n-1}^2(z_n) + J_{n+1}^2(z_n) - 2J_n^2(z_n), \\ g_n &\equiv g_n(z_n) = \frac{4n^2 J_n^2(z_n)}{z_n^2}, \\ h_n &\equiv h_n(z_n) = J_{n-1}^2(z_n) - J_{n+1}^2(z_n), \end{aligned} \quad (40)$$

where $J_n(z)$ is the Bessel function. Functions (40) depend on x , y and ξ via the single argument

$$z_n = \frac{\xi}{\sqrt{1+\xi^2}} n s_n. \quad (41)$$

For small value of this argument one has

$$f_n = g_n = h_n = \frac{(z_n/2)^{2(n-1)}}{[(n-1)!]^2} \quad \text{at } z_n \rightarrow 0, \quad (42)$$

in particular,

$$f_1 = g_1 = h_1 = 1 \quad \text{at } z_1 = 0. \quad (43)$$

It is useful to note that this argument is small for small ξ , as well as for small or for large values of y :

$$z_n \rightarrow 0 \quad \text{either at } \xi \rightarrow 0, \quad \text{or at } y \rightarrow y_n, \quad \text{or at } y \rightarrow 0. \quad (44)$$

The results of our calculations are the following. The item $F_0^{(n)}$, related to the total cross section (35), reads

$$\begin{aligned} F_0^{(n)} &= \left(\frac{1}{1-y} + 1-y \right) f_n - \frac{s_n^2}{1+\xi^2} g_n - \\ &\quad \left[\frac{y s_n}{\sqrt{1+\xi^2}} \zeta_2 - \frac{y(2-y)}{1-y} c_n \zeta_3 \right] h_n P_c. \end{aligned} \quad (45)$$

The polarization of the final photons $\xi_j^{(f)}$ is given by Eq. (36) where

$$\begin{aligned} F_1^{(n)} &= \frac{y}{1-y} \frac{s_n}{\sqrt{1+\xi^2}} h_n P_c \zeta_1, \\ F_2^{(n)} &= \left(\frac{1}{1-y} + 1-y \right) c_n h_n P_c - \frac{y s_n c_n}{\sqrt{1+\xi^2}} g_n \zeta_2 + \\ &\quad y \left(\frac{2-y}{1-y} f_n - \frac{s_n^2}{1+\xi^2} g_n \right) \zeta_3, \\ F_3^{(n)} &= 2(f_n - g_n) + s_n^2(1+\Delta) g_n - \frac{y}{1-y} \frac{s_n}{\sqrt{1+\xi^2}} h_n P_c \zeta_2, \end{aligned} \quad (46)$$

here we use the notation

$$\Delta = \frac{\xi^2}{1 + \xi^2}.$$

The polarization of the final electrons $\zeta_j^{(f)}$ is given by Eqs. (37), (38) with

$$\begin{aligned} G_1^{(n)} &= \langle G_1^{(n)} \rangle \zeta_1, \\ G_2^{(n)} &= \langle G_1^{(n)} \rangle \zeta_2 - \frac{y s_n}{(1-y)\sqrt{1+\xi^2}} (c_n g_n \zeta_3 + h_n P_c), \\ G_3^{(n)} &= \langle G_3^{(n)} \rangle + \frac{y s_n c_n}{\sqrt{1+\xi^2}} g_n \zeta_2, \end{aligned} \quad (47)$$

where we introduced the notations

$$\begin{aligned} \langle G_1^{(n)} \rangle &= 2f_n - \frac{s_n^2}{1+\xi^2} g_n, \\ \langle G_3^{(n)} \rangle &= \left[\left(\frac{1}{1-y} + 1-y \right) f_n - \left(1 + \frac{y^2}{1-y} \right) \frac{s_n^2}{1+\xi^2} g_n \right] \zeta_3 + \frac{y(2-y)c_n}{1-y} h_n P_c. \end{aligned} \quad (48)$$

At last, the correlation of the final particles' polarizations are

$$\begin{aligned} H_{11}^{(n)} &= \frac{y s_n}{\sqrt{1+\xi^2}} h_n P_c + \frac{y}{1-y} \left[(2-y)(f_n - g_n) + (1-y+\Delta) s_n^2 g_n \right] \zeta_2 - \frac{y c_n s_n}{\sqrt{1+\xi^2}} g_n \zeta_3, \\ H_{21}^{(n)} &= -\frac{y}{1-y} \left\{ (2-y)(f_n - g_n) + [1 + (1-y)\Delta] s_n^2 g_n \right\} \zeta_1, \\ H_{31}^{(n)} &= \frac{y c_n s_n}{(1-y)\sqrt{1+\xi^2}} g_n \zeta_1, \quad H_{12}^{(n)} = 2c_n h_n P_c \zeta_1, \\ H_{22}^{(n)} &= -\frac{y c_n s_n}{(1-y)\sqrt{1+\xi^2}} g_n + 2c_n h_n P_c \zeta_2 - \frac{y s_n}{(1-y)\sqrt{1+\xi^2}} h_n P_c \zeta_3, \\ H_{32}^{(n)} &= \frac{y}{1-y} \left[(2-y)f_n - \frac{s_n^2}{1+\xi^2} g_n \right] + \frac{y s_n}{\sqrt{1+\xi^2}} h_n P_c \zeta_2 + \frac{2-2y+y^2}{1-y} c_n h_n P_c \zeta_3, \\ H_{13}^{(n)} &= \left[\frac{2-2y+y^2}{1-y} (f_n - g_n) + \left(1 + \frac{1-y+y^2}{1-y} \Delta \right) s_n^2 g_n \right] \zeta_1, \\ H_{23}^{(n)} &= -\frac{y s_n}{\sqrt{1+\xi^2}} h_n P_c + \left[\frac{2-2y+y^2}{1-y} (f_n - g_n) + \left(\frac{1-y+y^2}{1-y} + \Delta \right) s_n^2 g_n \right] \zeta_2 + \\ &\quad \frac{y c_n s_n}{\sqrt{1+\xi^2}} g_n \zeta_3, \\ H_{33}^{(n)} &= -\frac{y c_n s_n}{(1-y)\sqrt{1+\xi^2}} g_n \zeta_2 + \left[2(f_n - g_n) + (1+\Delta) s_n^2 g_n \right] \zeta_3. \end{aligned} \quad (49)$$

4 Going to the collider system

4.1 Exact relations

As an example of the application of the above formulas, let us consider the nonlinear Compton scattering in the frame of reference, in which an electron performs a head-on collisions with a laser photons, i.e. in which $\mathbf{p} \parallel (-\mathbf{k})$. In what follows, we call it as the “collider system”. In this system we choose the z -axis along the initial electron

momentum \mathbf{p} . Azimuthal angles φ , β and β' of vectors \mathbf{k}' , $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}'$ are defined with respect to one fixed x -axis.

In such a system the unit vectors \mathbf{n}_j , defined in (28), has a simple form:

$$\mathbf{n}_1 = \frac{\mathbf{p} \times \mathbf{p}'}{|\mathbf{p} \times \mathbf{p}'|} = \frac{\mathbf{k} \times \mathbf{k}'}{|\mathbf{k} \times \mathbf{k}'|}, \quad \mathbf{n}_2 = \frac{\mathbf{p} \times \mathbf{n}_1}{|\mathbf{p} \times \mathbf{n}_1|} = \frac{\mathbf{k}'_{\perp}}{|\mathbf{k}'_{\perp}|}, \quad \mathbf{n}_3 = \frac{\mathbf{p}}{|\mathbf{p}|}, \quad (50)$$

and invariants ζ_j are equal to

$$\zeta_1 = \boldsymbol{\zeta} \mathbf{n}_1 = \zeta_{\perp} \sin(\varphi - \beta), \quad \zeta_2 = \boldsymbol{\zeta} \mathbf{n}_2 = \zeta_{\perp} \cos(\varphi - \beta), \quad \zeta_3 = \boldsymbol{\zeta} \mathbf{n}_3 = 2\lambda_e. \quad (51)$$

Here \mathbf{k}'_{\perp} and $\boldsymbol{\zeta}_{\perp}$ stands for the transverse components of the vectors \mathbf{k}' and $\boldsymbol{\zeta}$ with respect to the vector \mathbf{p} , and $\zeta_{\perp} = |\boldsymbol{\zeta}_{\perp}|$. Therefore, in this frame of reference, ζ_1 is the transverse polarization of the initial electron perpendicular to the scattering plane, ζ_2 is the transverse polarization in that plane and ζ_3 is the doubled mean helicity of the initial electron.

A phase volume element in (33) is equal to

$$d\Gamma = dy d\varphi, \quad (52)$$

and the differential cross section, summed over spin states of the final particles, is

$$\frac{d\sigma_n}{dy d\varphi} = \frac{r_e^2}{x} F_0^{(n)}. \quad (53)$$

Integrating this expression over φ , we find that (similar to the linear Compton scattering) the differential $d\sigma_n/dy$ and the total cross sections σ_n for a given harmonic n do not depend on the the transverse polarization of the initial electron :

$$\begin{aligned} \frac{d\sigma_n}{dy} &= \frac{2\pi r_e^2}{x} \langle F_0^{(n)} \rangle, \quad \sigma_n = \frac{2\pi r_e^2}{x} \int_0^{y_n} \langle F_0^{(n)} \rangle dy, \\ \langle F_0^{(n)} \rangle &= \left(\frac{1}{1-y} + 1-y \right) f_n - \frac{s_n^2}{1+\xi^2} g_n + \frac{y(2-y)}{1-y} c_n h_n \zeta_3 P_c. \end{aligned} \quad (54)$$

The polarization of the final photon is given by Eqs. (36). The polarization vector of the final electron is determined by Eq. (38), but unfortunately, the unit vectors \mathbf{n}'_j in this equation has no simple form similar to (50). Therefore, we have to find the characteristics that are usually used for description of the electron polarization — the helicity of the final electron λ'_e and its transverse (to the vector \mathbf{p}') polarization $\boldsymbol{\zeta}'_{\perp}$ in the considered collider system. For this purpose we introduce unit vectors $\boldsymbol{\nu}_j$, one of them is directed along the momentum of the final electron \mathbf{p}' and two others are in the transverse plane (compare with Eq. (50)):

$$\boldsymbol{\nu}_1 = \mathbf{n}_1, \quad \boldsymbol{\nu}_2 = \frac{\mathbf{p}' \times \mathbf{n}_1}{|\mathbf{p}' \times \mathbf{n}_1|}, \quad \boldsymbol{\nu}_3 = \frac{\mathbf{p}'}{|\mathbf{p}'|}; \quad \boldsymbol{\nu}_i \boldsymbol{\nu}_j = \delta_{ij}. \quad (55)$$

After that one has

$$\boldsymbol{\zeta}' \boldsymbol{\nu}_1 = \zeta'_{\perp} \sin(\varphi - \beta'), \quad \boldsymbol{\zeta}' \boldsymbol{\nu}_2 = \zeta'_{\perp} \cos(\varphi - \beta'), \quad \boldsymbol{\zeta}' \boldsymbol{\nu}_3 = 2\lambda'_e. \quad (56)$$

Therefore, $\zeta' \nu_1$ is the transverse polarization of the final electron perpendicular to the scattering plane, $\zeta' \nu_2$ is the transverse polarization in that plane and $\zeta' \nu_3$ is the doubled mean helicity of the final electron.

Let us discuss the relation between the projection $\zeta' \nu_j$, defined above, and the invariants ζ'_j , defined in (26). In the collider system the vectors ν_1 and \mathbf{n}'_1 coincide

$$\nu_1 = \mathbf{n}'_1, \quad (57)$$

therefore,

$$\zeta' \nu_1 = \zeta'_1. \quad (58)$$

Two other unit vectors \mathbf{n}'_2 and \mathbf{n}'_3 are in the scattering plane and they can be obtained from vectors ν_2 and ν_3 by the rotation around the axis ν_1 on the angle $(-\Delta\theta)$:

$$\zeta' \nu_2 = \zeta'_2 \cos \Delta\theta + \zeta'_3 \sin \Delta\theta, \quad \zeta' \nu_3 = \zeta'_3 \cos \Delta\theta - \zeta'_2 \sin \Delta\theta, \quad (59)$$

where

$$\cos \Delta\theta = \nu_3 \mathbf{n}'_3 \quad \sin \Delta\theta = -\nu_3 \mathbf{n}'_2. \quad (60)$$

As a result, the polarization vector of the final electron (38) is expressed as follows

$$\zeta^{(f)} = \nu_1 \frac{G_1}{F_0} + \nu_2 \left(\frac{G_2}{F_0} \cos \Delta\theta + \frac{G_3}{F_0} \sin \Delta\theta \right) + \nu_3 \left(\frac{G_3}{F_0} \cos \Delta\theta - \frac{G_2}{F_0} \sin \Delta\theta \right). \quad (61)$$

4.2 Approximate formulas

All the above formulas are exact. In this subsection we give some approximate formulas useful for application to the important case of high-energy $\gamma\gamma$ and γe colliders. It is expected that in the conversion region of these colliders, an electron with the energy $E \sim 100$ GeV performs a head-on collisions with a laser photons having the energy $\omega \sim 1$ eV per a single photon [14]. In this case the most important kinematical range corresponds to almost back-scattered final photons, i. e. the initial electron is ultra-relativistic and the final photon is emitted at small angle θ_γ with respect to the z -axis:

$$E \gg m, \quad \theta_\gamma \ll 1. \quad (62)$$

In this approximation we have

$$x \approx \frac{4E\omega}{m^2}, \quad y \approx \frac{\omega'}{E} \approx 1 - \frac{E'}{E}, \quad (63)$$

therefore, Eq. (53) gives us the distribution of the final photons over the energy and the azimuthal angle. Besides, the photon emission angle for reaction (2) is

$$\theta_\gamma \approx \frac{m}{E} \sqrt{nx + 1 + \xi^2} \sqrt{\frac{y_n}{y} - 1} \quad (64)$$

and $\theta_\gamma \rightarrow 0$ at $y \rightarrow y_n$. For a given y , the photon emission angle increases with the increase of n . The electron scattering angle is small

$$\theta_e \approx \frac{y\theta_\gamma}{1-y} \leq \frac{2n}{\sqrt{1+\xi^2}} \frac{\omega}{m}. \quad (65)$$

It is not difficult to check that, for the considered case, the angle $\Delta\theta$ between vector $\boldsymbol{\nu}_3$ and \mathbf{n}'_3 is very small:

$$\Delta\theta \approx |\boldsymbol{\nu}_{3\perp} - \mathbf{n}'_{3\perp}| \approx \frac{m\theta_e}{2E'} \leq \frac{n}{\sqrt{1+\xi^2}} \frac{\omega}{E'}. \quad (66)$$

It means that invariants ζ'_j from (26) almost coincide with projections $\boldsymbol{\zeta}' \cdot \boldsymbol{\nu}_j$, defined in (56), and that the exact equation (61) for the polarization of the final electron can be replaced with a high accuracy by the approximate equation

$$\boldsymbol{\zeta}^{(f)} \approx \sum_{j=1}^3 \frac{G_j}{F_0} \boldsymbol{\nu}_j. \quad (67)$$

5 Summary and comparison with other papers

Our main result is given by Eqs. (45)–(49) which present 16 functions F_0 , F_j , G_j and H_{ij} with $i, j = 1 \div 3$. They describe completely all polarization properties of the nonlinear Compton scattering in a rather compact form.

In the literature we found 5 functions which can be compared with ours F_0 , F_2 and G_j . Functions F_0 and F_j enter the total cross section (35), (54), the differential cross sections (53), (54) and the Stokes parameters of the final photons (36). Two of these functions were calculated in [8] (the function F_0 only at $\zeta_j = 0$), in [9, 10, 11] (the functions F_0 and F_2 only at $\zeta_1 = \zeta_2 = 0$) and in [12] (the function F_0 for arbitrary ζ_j). For these cases our results coincide with the above mentioned ones.

The polarization of the final electrons is described by functions G_j (47), (48). They enter the polarization vector $\boldsymbol{\zeta}^{(f)}$ given by exact (38), (61) and approximate (67) equations. Here our results differ slightly from those in [11, 12]. Namely, function G_2 at $\zeta_1 = \zeta_2 = 0$ given in [11] and functions G_j for arbitrary ζ_j obtained in [12] coincide with ours. However, the polarization vector $\boldsymbol{\zeta}^{(f)}$ in the collider system is obtained in papers [11, 12] only in the approximate form equivalent to our approximate Eq. (67).

The correlation of the final particles' polarizations are described by functions H_{ij} given in Eqs. (49). We did not find in the literature any on these functions.

At small ξ^2 all harmonics with $n > 1$ disappear due to properties (42) and (44),

$$d\sigma_n(\xi'_i, \xi'_j) \propto \xi^{2(n-1)} \quad \text{at} \quad \xi^2 \rightarrow 0. \quad (68)$$

We checked that in this limit our expression for $d\sigma(\xi'_i, \xi'_j)$ coincides with the result known for the linear Compton effect, see Appendix.

Acknowledgements

We are grateful to I. Ginzburg, M. Galynskii, A. Milshtein, S. Polityko and V. Telnov for useful discussions. This work is partly supported by INTAS (code 00-00679) and RFBR (code 03-02-17734); D.Yu.I. acknowledges the support of Alexander von Humboldt Foundation.

Appendix: Limit of the weak laser field

At $\xi^2 \rightarrow 0$, the cross section (33) has the form

$$d\sigma(\xi'_i, \xi'_j) = \frac{r_e^2}{4x} F d\Gamma; \quad d\Gamma = \delta(p + k - p - k') \frac{d^3 k'}{\omega'} \frac{d^3 p'}{E'}, \quad (69)$$

where

$$F = F_0 + \sum_{j=1}^3 (F_j \xi'_j + G_j \xi'_j) + \sum_{i,j=1}^3 H_{ij} \xi'_i \xi'_j. \quad (70)$$

To compare it with the cross section for the linear Compton scattering, we should take into account that the Stokes parameters of the initial photon have values (20) and that our invariants c_1 and s_1 transforms at $\xi^2 = 0$ to

$$c = 1 - 2r, \quad s = 2\sqrt{r(1-r)} \quad (71)$$

with $r = y/[x(1-y)]$. Our functions

$$\begin{aligned} F_0 &= \frac{1}{1-y} + 1 - y - s^2 - y \left(s \zeta_2 - \frac{2-y}{1-y} c \zeta_3 \right) P_c, \quad F_1 = \frac{y}{1-y} s P_c \zeta_1, \\ F_2 &= \left(\frac{1}{1-y} + 1 - y \right) c P_c - y s c \zeta_2 + y \left(\frac{2-y}{1-y} - s^2 \right) \zeta_3, \\ F_3 &= s^2 - \frac{y}{1-y} s P_c \zeta_2 \end{aligned} \quad (72)$$

coincide with those in [3]. Our functions

$$\begin{aligned} G_1 &= (1 + c^2) \zeta_1, \quad G_2 = -\frac{ys}{1-y} P_c + (1 + c^2) \zeta_2 - \frac{y s c}{1-y} \zeta_3, \\ G_3 &= y \frac{2-y}{1-y} c P_c + y s c \zeta_2 + \left(1 + \frac{1-y+y^2}{1-y} c^2 \right) \zeta_3. \end{aligned} \quad (73)$$

coincide with functions Φ_j given by Eqs. (31) in [4]. At last, we check that our functions

$$\begin{aligned} H_{11} &= y s P_c + y s^2 \zeta_2 - y c s \zeta_3, \quad H_{21} = -\frac{y}{1-y} s^2 \zeta_1, \\ H_{31} &= \frac{y c s}{1-y} \zeta_1, \quad H_{12} = 2c P_c \zeta_1, \\ H_{22} &= -\frac{y c s}{1-y} + 2c P_c \zeta_2 - \frac{y s}{1-y} P_c \zeta_3, \\ H_{32} &= \frac{y}{1-y} (2 - y - s^2) + y s P_c \zeta_2 + \frac{2 - 2y + y^2}{1-y} c P_c \zeta_3, \\ H_{13} &= s^2 \zeta_1, \quad H_{23} = -y s P_c + \frac{1-y+y^2}{1-y} s^2 \zeta_2 + y c s \zeta_3, \\ H_{33} &= -\frac{y c s}{1-y} \zeta_2 + s^2 \zeta_3 \end{aligned} \quad (74)$$

coincide with the corresponding functions from [2]. At such a comparison, one needs to take into account that the set of unit 4-vectors, used in our paper (see Eqs. (19), (23)),

and one used in paper [2] are different. The relations between our notations and the notations used in [2], which are marked below by super-index G , are the following:

$$\begin{aligned}\xi_{1,3}^G &= -\xi_{1,3}, \quad \xi_2^G = \xi_2, \quad \xi_{1,3}'^G = -\xi_{1,3}', \quad \xi_2'^G = \xi_2', \\ \zeta_{1,3}^G &= \zeta_{3,1}, \quad \zeta_2^G = -\zeta_2, \quad \zeta_1'^G = c\zeta_3' - s\zeta_2', \quad \zeta_2'^G = -c\zeta_2' - s\zeta_3', \quad \zeta_3'^G = \zeta_1'.\end{aligned}\tag{75}$$

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